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A MODULE AS A TORSION-FREE COVER

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ABSTRACT

Let R be a ring, and let $(\mathcal{T}, \mathcal{F})$ be an hereditary torsion theory of left R-modules. An epimorphism $\psi : M \to X$ is called a torsion-free cover of X if (1) $M \in \mathcal{F}$, (2) every homomorphism from a torsion-free module into X can be factored through M, and (3) ker ψ contains no nonzero $\mathcal{T}$-closed submodules of M. Conditions on M and N are studied to determine when the natural maps $M \to M/N$ and $Q(M) \to Q(M)/N$ are torsion-free covers, when $Q(M)$ is the localization of M with respect to $(\mathcal{T}, \mathcal{F})$. If $M \to M/N$ is a torsion-free cover and M is projective, then $N \subseteq \operatorname{rad} M$. Consequently, the concepts of projective cover and torsion-free cover coincide in some interesting cases.

Let R be a ring with identity, and let ${}_R\mathcal{M}$ be the category of unital left R-modules. Let $(\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on ${}_R\mathcal{M}$ with filter of left ideals $\mathcal{L}$. Let R be torsion free and let Q be the quotient ring of R with respect to $(\mathcal{T}, \mathcal{F})$. If $X \in {}_R\mathcal{M}$, let $E(X)$ denote an injective hull of X, and $Q(X)$ the quotient module of X with respect to $(\mathcal{T}, \mathcal{F})$. For all notions concerning torsion theories and other undefined terms, we refer the reader to [4].

Throughout this paper let M be a torsion-free module with submodule N and canonical epimorphism $\pi : M \to M/N$. Let $\pi^* : \operatorname{Hom}_R(Q, M) \to \operatorname{Hom}_R(Q, M/N)$ be the canonical R (and Q) homomorphism. The mapping $\pi : M \to M/N$ is called a *precover* for M/N if any $f \in \operatorname{Hom}(X, M/N)$, where $X \in \mathcal{F}$, can be lifted canonically to $\operatorname{Hom}(X, M)$. The precover is a *torsion-free cover* if, in addition, N contains no nonzero $\mathcal{T}$-closed submodules of M. In order to eliminate trivial cases we will henceforth assume that $M/N \notin \mathcal{F}$.

The existence of torsion-free covers for modules over a commutative integral domain was first examined by Enochs [3]. The existence of torsion-free covers for abstract torsion theories over more general rings is the subject of a number of papers including [5] and [7]. However, the problem of determining when a module is a torsion-free cover of another remains a distinct problem, which was

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first investigated by Banachewski [1]. For perfect torsion theories, Banachewski described the torsion-free cover of a module X as the evaluation map applied to a certain submodule of $\mathrm{Hom}_R(Q, E(X))$. We wish to study the more accessible situation where the canonical map $M \to M/N$ is a torsion-free cover. Recently, Cheatham [2] and Matlis [6] have obtained results about when a natural map from a quotient field or a commutative integral domain can be the cover of a module. In this paper we generalize a result of [2] and the main result of [6] to more general types of modules in the setting of an abstract torsion theory. (See Theorem 1 and Corollaries 4 and 5.) This enables us to obtain a result relating torsion-free covers to projective covers. Since the proofs in [2] and [6] rely heavily on properties of commutative rings, our proofs must necessarily be quite different from the previous ones; fortunately, the proofs of our key results (Theorems 2 and 3) are considerably shorter than the corresponding proofs in [6].

For the sake of easy reference, we include the following folk theorems, whose proofs are easy modifications of published results.

THEOREM A. *If E, E', and E'' are R-modules with $\psi : E \to E''$ and $\psi' : E' \to E''$ torsion-free covers and if $f : E \to E'$ satisfies $f\psi' = \psi$, then f is an isomorphism.*

PROOF. See the proof of [3, theorem 2].

THEOREM B. *Suppose that $X \in {}_R\mathcal{M}$ and that $(\mathcal{T}, \mathcal{F})$ is perfect. Let $H = \mathrm{Hom}_R(Q, E(X))$ and $C(X) = \{f \in H \mid (1)f \in X\}$. Define $\phi : C(X) \to X$ by $(f)\phi = (1)f$. Then $\phi : C(X) \to X$ is a torsion-free cover of X.*

PROOF. See [1] or [5, page 247].

We are now ready to give a generalization of [2, theorem 1].

THEOREM 1. *Suppose that $(\mathcal{T}, \mathcal{F})$ has exact localization, that M is $\mathcal{T}$-injective, and that $M/N \notin \mathcal{F}$. Then the following statements are equivalent.*

(1) *$\pi : M \to M/N$ is a torsion-free cover.*

(2) *M/N is $\mathcal{T}$-injective and π^* is an isomorphism.*

(3) *M/N is $\mathcal{T}$-injective, $\mathrm{Hom}_R(Q, N) = 0$, and $\mathrm{Ext}_R(Q, N) = 0$.*

(4) *$\mathrm{Ext}_R(X, N) = 0$ for all $X \in \mathcal{F}$ and $\mathrm{Hom}_R(Q, N) = 0$.*

PROOF. (1) $\Rightarrow$ (2). If $I \in \mathcal{L}$ and $f : I \to M/N$, there exists $g : I \to M$ such that $g\pi = f$. There exists $h : R \to M$ such that $h|_I = g$. Then $h\pi$ extends f; so M/N is $\mathcal{T}$-injective. By the definition of torsion-free cover we have the following exact sequence:

$$0 \longrightarrow \operatorname{Hom}(Q,N) \rightarrow \operatorname{Hom}(Q,M) \xrightarrow{\pi^*} \operatorname{Hom}(Q,M/N) \longrightarrow 0.$$

If $0 \neq f \in \operatorname{Hom}(Q,N)$, then $\operatorname{Im} f$, being a torsion-free image of a torsion-free $\mathcal{T}$-injective module, is a torsion-free $\mathcal{T}$-injective module [4, prop. 16.1]. Hence $\operatorname{Im} f$ is $\mathcal{T}$-closed in M; so $f = 0$. Hence $\operatorname{Hom}(Q,N) = 0$ and π^* is an isomorphism.

(2) ⇒ (3). Since π^* is an isomorphism, we have $\operatorname{Hom}(Q,N) = 0$ and

$$0 \longrightarrow \operatorname{Hom}(Q,M) \xrightarrow{\pi^*} \operatorname{Hom}(Q,M/N) \longrightarrow \operatorname{Ext}(Q,N) \longrightarrow \operatorname{Ext}(Q,M)$$

is exact. Since M is $\mathcal{T}$-injective, we also have the exact sequence

$$0 = \operatorname{Ext}(Q/R,M) \rightarrow \operatorname{Ext}(Q,M) \rightarrow \operatorname{Ext}(R,M) = 0.$$

Hence $\operatorname{Ext}(Q,M) = 0$, and it follows that $\operatorname{Ext}(Q,N) = 0$.

(3) ⇒ (4). If $X \in \mathcal{F}$, then we have the exact sequence:

$$\operatorname{Ext}(Q(X),N) \rightarrow \operatorname{Ext}(X,N) \rightarrow \operatorname{Ext}^2(Q(X)/X,N).$$

If $0 \rightarrow L \rightarrow \bigoplus Q \rightarrow Q(X) \rightarrow 0$ is a free resolution of $Q(X)$ as a Q-module, then L is a Q-module and we have the exact sequence:

$$0 = \operatorname{Hom}(\bigoplus Q,N) \rightarrow \operatorname{Hom}(L,N) \rightarrow \operatorname{Ext}(Q(X),N) \rightarrow \operatorname{Ext}(\bigoplus Q,N) = 0.$$

Since L is a Q-module, then $\operatorname{Hom}(L,N) = 0$ by (3), and hence $\operatorname{Ext}(Q(X),N) = 0$.

Since $Q(X)/X \in \mathcal{T}$, we have by the exactness of the localizing functor of $(\mathcal{T},\mathcal{F})$ and [4, prop. 16.1] that $\operatorname{Ext}^2(Q(X)/X,M) = 0$. Hence the exactness of

$$0 = \operatorname{Ext}(Q(X)/X,M/N) \rightarrow \operatorname{Ext}^2(Q(X)/X,N) \rightarrow \operatorname{Ext}^2(Q(X)/X,M) = 0$$

implies that $\operatorname{Ext}^2(Q(X)/X,N) = 0$. Hence $\operatorname{Ext}(X,N) = 0$.

(4) ⇒ (1). If $X \in \mathcal{F}$, then

$$\operatorname{Hom}(X,M) \rightarrow \operatorname{Hom}(X,M/N) \rightarrow \operatorname{Ext}(X,N) = 0$$

and hence maps from X to M/N can be factored through M.

If C is contained in N and C is a nonzero $\mathcal{T}$-closed submodule of M, then $Q(C) = C$ and C is a left Q-module. Hence $0 \neq \operatorname{Hom}(Q,C) \subseteq \operatorname{Hom}(Q,N)$, which is impossible. Hence N contains no $\mathcal{T}$-closed submodules.

Our next result gives a key property of $E(M/N)$ in the case where $M \rightarrow M/N$ is a torsion-free cover.

THEOREM 2. *If $\pi : M \to M/N$ is a torsion-free cover, then $E(M)/N$ is an essential extension of M/N.*

PROOF. Suppose that $0 \neq e + N \in E(M)/N$ with $R(e+N) \cap (M/N) = 0$. It follows that $Re \cap M \subseteq N$ and that the epimorphism $\pi_1 : Re + M \to M/N$ given by $(re+m)\pi_1 = m + N$ $(r \in R,\ m \in M)$ is well-defined with $\ker \pi_1 = Re + N$.

If $i : M \to Re + M$ is the inclusion map, then clearly $i\pi_1 = \pi$. If $X \in \mathcal{F}$ and $g : X \to M/N$, there exists $h : X \to M$ such that $h\pi = g$. Then $hi\pi_1 = h\pi = g$; so hi lifts g to $M + Re$. If C is a submodule of $\ker \pi_1$ that is $\mathcal{T}$-closed in $Re + M$, then $C \cap M$ is $\mathcal{T}$-closed in M. But

$$C \cap M \subseteq \ker \pi_1 \cap M = (Re + N) \cap M \subseteq N.$$

Hence $C = 0$; and so $\pi_1 : Re + M \to M/N$ is a torsion-free cover. Since $i\pi_1 = \pi$, we have by Theorem A that i is an isomorphism. Hence $Re + M = M$ and $e \in M$, which is contrary to the assumption.

THEOREM 3. *Let $(\mathcal{T}, \mathcal{F})$ be a perfect torsion theory. If $\pi : M \to M/N$ is a torsion-free cover, then $Q(M) \to Q(M)/N$ is a torsion-free cover.*

PROOF. By Theorem 2 we may assume that $E(M/N) = E(Q(M)/N)$. Let $H = \mathrm{Hom}(Q, E(M/N))$ and $C = \{f \in H \mid (1)f \in Q(M)/N\}$.

If $x \in Q(M)$, define $h_x \in H$ by $(q)h_x = qx + N$. Since N has no nonzero Q-submodules, we may assume that $Q(M) \subseteq C$ via the correspondence $x \to h_x$. By Theorem B we have that C is a torsion-free cover of $Q(M)/N$, and we may identify M with $\{f \in H \mid (1)f \in M/N\}$.

If $f \in C \setminus M$, then $(1)f \in Q(M)/N$ and there exists $J \in \mathcal{L}$ such that $J((1)f) \subseteq M/N$. If $r \in J$, then using the R-module structure on H and F, we have

$$(1)(rf) = (r)f = r((1)f) \in M/N.$$

Thus, under the isomorphism, $rf \in M$, and we have $C/M \in \mathcal{T}$. Since $f \notin M$, $0 \neq (1)f \in Q(M)/N$. Since M/N is essential in $Q(M)/N$ by Theorem 2, there exists $r \in R$ with $0 \neq r((1)f) = (1)(rf) \in M/N$. Hence $0 \neq rf \in M$ and M is essential in C. We conclude that $M \subseteq Q(M) \subseteq C \subseteq E(M)$ with $C/M \in \mathcal{T}$. Hence $C = Q(M)$ and $Q(M) \to Q(M)/N$ is a torsion-free cover.

Our next two corollaries are generalizations of the main result (Theorem 1) of [6].

COROLLARY 4. *Suppose that N contains no nonzero $\mathcal{T}$-closed submodules of M. If $(\mathcal{T}, \mathcal{F})$ is perfect, then the following statements are equivalent.*

(1) $M \to M/N$ *is a torsion-free cover.*

(2) $Q(M) \to Q(M)/N$ *is a torsion-free cover.*

(3) $\mathrm{Ext}_R(X, N) = 0$ *for all* $X \in \mathcal{F}$.

PROOF. (2) $\Leftrightarrow$ (3). The standing hypothesis gives that $\mathrm{Hom}(Q, N) = 0$; so this equivalence follows from Theorem 1.

(1) $\Rightarrow$ (2) is Theorem 3.

(3) $\Rightarrow$ (1) is clear.

COROLLARY 5. *If* $(\mathcal{T}, \mathcal{F})$ *is perfect, then the following statements are equivalent.*

(1) $\pi : M \to M/N$ *is a torsion-free cover.*

(2) $\pi : Q(M) \to Q(M)/N$ *is a torsion-free cover and* $Q(M)/N$ *is the* $\mathcal{T}$*-injective hull of* M/N.

PROOF. (1) $\Rightarrow$ (2). The first part follows from Theorem 3. The second part follows from Theorem 2 and [5, prop. 2.4].

(2) $\Rightarrow$ (1). By Theorem 1 we have $\mathrm{Ext}(X, N) = 0$ for all $X \in \mathcal{F}$; and hence $\pi : M \to M/N$ is a precover. If $C \subseteq N$ is a nonzero $\mathcal{T}$-closed submodule of M, then C is not $\mathcal{T}$-closed in $Q(M)$. Now $Q(C)$ is the $\mathcal{T}$-closure of C in $Q(M)$, $Q(C) \neq C$, and $Q(C) \cap M = C$. Thus $(Q(C)/N) \cap (M/N) = 0$. Since M/N is essential in $Q(M)/N$, we have a contradiction.

COROLLARY 6. *If* $(\mathcal{T}, \mathcal{F})$ *is perfect and* $\pi : M \to M/N$ *is a torsion-free cover, then the following statements hold.*

(1) $\mathrm{Hom}_R(Q, N) = 0$.

(2) $\mathrm{Ext}_R(N, X) = 0$ *for all* $X \in \mathcal{F}$.

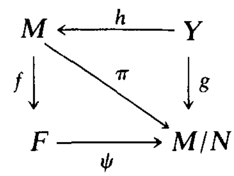
(3) $E(M/N) = E(M)/N$.

PROOF. By Theorems 3 and 1 we have (1) and (2). By an obvious modification of the proof that $Q(M)/N$ is $\mathcal{T}$-injective, we see that $E(M)/N$ is injective.

THEOREM 7. *If* $\pi : M \to M/N$ *is a torsion-free cover and* M *is projective, then* $N \subseteq \mathrm{rad}\, M$.

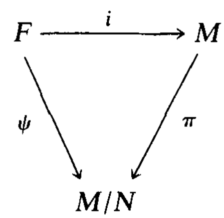
PROOF. The result is trivial if $M = \mathrm{rad}\, M$; so suppose there exists a maximal submodule X of M with $N \not\subseteq X$. Then $M = N + X$, $M/N \cong X/(X \cap N)$, and we have a natural epimorphism $\theta : X \to M/N$. Since M is projective, there exists $f : M \to X$ such that $f\theta = \pi$. Since $\ker f \subseteq \ker \pi$, we have that f is a monomorphism.

Let $F = \mathrm{Im}\, f$ and $\psi = \theta\,|_F$. We will show that F and ψ provide a torsion-free cover for M/N. Suppose that $g: Y \to M/N$, where $Y \in \mathcal{F}$. Then there exists $h: Y \to M$, and we have the following commutative diagram:



Then $hf\psi = h\pi = g$; so hf lifts g to F. Since ψ is onto, there are no $\mathcal{T}$-closed submodules of F contained in $\ker \psi$ (as $(\ker \psi)f^{-1} = N$). Hence F is a torsion-free cover of M/N.

Consider the inclusion map $i: F \to M$ and the following commutative diagram:



By Theorem A, i is an isomorphism; so $F = M$. Since $F \subsetneq X$, we have a contradiction.

Theorem 7 enables us to relate torsion-free covers to projective covers in the next result. In particular, it shows that the torsion-free cover $R \to R/I$ studied by Matlis [6] for integral domains is actually a projective cover of R/I.

Corollary 8. *If M is projective, if every proper submodule of M is contained in a maximal submodule (for example, if M is finitely generated), and if $\pi: M \to M/N$ is a torsion-free cover, then $\pi: M \to M/N$ is a projective cover.*

A module has finite (Goldie) dimension if it contains no infinite direct sums of nonzero modules.

Theorem 9. *Suppose that M is finite dimensional. If $\pi: M \to M/N$ is a torsion-free cover and $\alpha: M \to M/N$ is a precover, then $\alpha: M \to M/N$ is a torsion-free cover.*

Proof. There exist mappings f and g such that the following diagram commutes:

$$\begin{array}{ccc} M & \underset{g}{\overset{f}{\rightleftarrows}} & M \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & M/N & \end{array}$$

Since $f\pi = \alpha$ and $g\alpha = \pi$, we have $\pi = g\alpha = gf\pi$. By Theorem A we have that gf is an isomorphism. Let $\beta = (gf)^{-1}$. Then $1 = gf\beta$ and $f\beta\pi = f\pi = \alpha$. Hence by replacing f by $f\beta$, we may assume that $f\pi = \alpha$, $g\alpha = \pi$, and $gf = 1$. Let $e = fg$. Then $e^2 = e$, and $M \cong \operatorname{Im} e \oplus \ker e = \operatorname{Im} g \oplus \ker f \cong M \oplus \ker f$. Since M is finite dimensional, we must have $\ker f = 0$. Thus f is an isomorphism, and the result follows.

Our final result extends [6, prop. 1].

COROLLARY 10. *Suppose that M is a finite dimensional, projective module and that $\pi : M \to M/N$ is a torsion-free cover.*

(1) *If $\alpha : M \to M/N$ is an epimorphism, then $\alpha : M \to M/N$ is a torsion-free cover.*

(2) *If $(\mathcal{T}, \mathcal{F})$ is perfect and $\alpha : Q(M) \to Q(M)/N$ is an epimorphism, then $\alpha : Q(M) \to Q(M)/N$ is a torsion-free cover.*

PROOF. The projectivity of M gives that α is a precover; so (1) follows from Theorem 9. For (2) we have the following (not necessarily commutative) diagram:

$$\begin{array}{ccc} Q(M) & \underset{g}{\overset{f}{\rightleftarrows}} & Q(M) \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & Q(M)/N & \end{array}$$

The mapping f is obtained by Theorem 3, and g is obtained by extending the map given by the projectivity of M. We have $g\alpha = \pi$ on M and $f\pi = \alpha$. If $m \in M$, we have $m\pi = mg\alpha = mgf\pi$; and so $mgf \in M$. Hence by Theorem A, gf is an isomorphism on M. Since $Q(M)$ is the $\mathcal{T}$-injective hull of M, then gf is also an isomorphism on $Q(M)$. If $\beta = (gf)^{-1}$, we have $gf\beta = 1$; as in the proof of Theorem 9, f is an isomorphism.

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